

Tripartite Bell inequality, random matrices and trilinear forms

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Abstract

In this seminar report, we present in detail the proof of a recent result due to J. Briët and T. Vidick, improving an estimate in a 2008 paper by D. Pérez-García, M. Wolf, C. Palazuelos, I. Villanueva, and M. Junge, estimating the growth of the deviation in the tripartite Bell inequality. The proof requires a delicate estimate of the norms of certain trilinear (or d -linear) forms on Hilbert space with coefficients in the second Gaussian Wiener chaos. Let E_{\vee}^n (resp. E_{\min}^n) denote $\ell_1^n \otimes \ell_1^n \otimes \ell_1^n$ equipped with the injective (resp. minimal) tensor norm. Here ℓ_1^n is equipped with its maximal operator space structure. The Briët-Vidick method yields that the identity map I_n satisfies (for some $c > 0$) $\|I_n : E_{\vee}^n \rightarrow E_{\min}^n\| \geq cn^{1/4}(\log n)^{-3/2}$. Let S_2^n denote the (Hilbert) space of $n \times n$ -matrices equipped with the Hilbert-Schmidt norm. While a lower bound closer to $n^{1/2}$ is still open, their method produces an interesting, asymptotically almost sharp, related estimate for the map $J_n : S_2^n \overset{\vee}{\otimes} S_2^n \overset{\vee}{\otimes} S_2^n \rightarrow \ell_2^{n^3} \overset{\vee}{\otimes} \ell_2^{n^3}$ taking $e_{i,j} \otimes e_{k,l} \otimes e_{m,n}$ to $e_{[i,k,m],[j,l,n]}$.

1. Tripartite Bell inequality

We will prove the following theorem due to J. Briët and T. Vidick, improving an estimate in Junge et al. The proof in [1] was kindly explained to me in detail by T. Vidick. The improvements below (improving the power of the logarithmic term) are routine refinements of the ideas in [1].

Theorem 1.1. *Let $Y^{(N)}$ be $N \times N$ a Gaussian random matrix with Gaussian entries all i.i.d. of mean zero and L_2 -norm equal to $N^{-1/2}$. Let $Y_j^{(N)}$ ($j = 1, 2, \dots$) be a sequence of independent copies of $Y^{(N)}$. There is a constant C such that with large probability we have for all scalars a_{ij} with $1 \leq i, j \leq N$*

$$\left\| \sum_{i,i'=1}^N a_{ii'} Y_i^{(N)} \otimes Y_{i'}^{(N)} \right\| \leq C(\log N)^{3/2} \left(\sum_{i,i'=1}^N |a_{ii'}|^2 \right)^{1/2}$$

Equivalently, let $g_{i,k,l}$ be i.i.d. Gaussian normal random variables, with $i, k, l \leq N$. Let $g'_{i,k,l}$ be an independent copy of the family $g_{i,k,l}$. Then the norm $\|\mathcal{T}\|_{\vee}$ of the tensor

$$\mathcal{T} = \sum g_{i,k,l} g'_{i',k',l'} e_{ii'} \otimes e_{kk'} \otimes e_{ll'}$$

in the triple injective tensor product $\ell_2^{N^2} \overset{\vee}{\otimes} \ell_2^{N^2} \overset{\vee}{\otimes} \ell_2^{N^2}$ satisfies for some C

$$\mathbb{E} \|\mathcal{T}\|_{\vee} \leq C(\log N)^{3/2} N$$

Remark 1.2. Note that if we replace in \mathcal{T} the random coordinates by a family of i.i.d. Gaussian normal variables indexed by N^6 , then by well known estimates (e.g. the Chevet inequality) the corresponding random tensor, denoted by $\hat{\mathcal{T}}$, satisfies $\mathbb{E} \|\hat{\mathcal{T}}\|_{\vee} \leq CN$.

Remark 1.3. Let g be a Gaussian vector in a finite dimensional (real) Hilbert space H and let g' be an independent copy of g . We assume (for simplicity) that the distribution of g and g' is the canonical Gaussian measure on H . Let u_i ($i = 1, \dots, M$) be operators on H . Let $Z_i = \langle u_i g, u_i g' \rangle$, and let $\hat{Z}_i = \langle u_i g, u_i g \rangle - \mathbb{E} \langle u_i g, u_i g \rangle$. We have then for any $p \geq 1$

$$(1.1) \quad 2^{-1} \left\| \sup_{i \leq M} |\hat{Z}_i| \right\|_p \leq \left\| \sup_{i \leq M} |Z_i| \right\|_p \leq \left\| \sup_{i \leq M} |\hat{Z}_i| \right\|_p,$$

and hence

$$(1.2) \quad \left\| \sup_{i \leq M} \langle u_i g, u_i g \rangle \right\|_p \leq 2 \left\| \sup_{i \leq M} |Z_i| \right\|_p + \sup_{i \leq M} \|u_i\|_{S_2}^2,$$

where $\|\cdot\|_{S_2}$ denotes the Hilbert-Schmidt norm. Indeed, (denoting \approx equality in distribution) we have

$$(g, g') \approx (2^{-1/2}(g + g'), 2^{-1/2}(g - g')).$$

Therefore

$$\langle u_i g, u_i g \rangle - \langle u_i g', u_i g' \rangle \approx 2 \langle u_i g, u_i g' \rangle.$$

Thus, if $\{\hat{Z}'_i\}$ is an independent copy of $\{\hat{Z}_i\}$, we have

$$\hat{Z}_i - \hat{Z}'_i \approx 2Z_i.$$

From this (1.1) follows easily and (1.2) is an immediate consequence.

Remark 1.4. Let us denote by $S_{2,1}(H)$ the class of operators u on H such that the eigenvalues λ_j of $|u|$ (rearranged as usual with multiplicity in non-increasing order) satisfy

$$\sum j^{-1/2} \lambda_j < \infty,$$

equipped with the quasi-norm (equivalent to a norm)

$$\|u\|_{2,1} = \sum j^{-1/2} \lambda_j < \infty.$$

It is clear that by Cauchy-Schwarz (for some constant C)

$$(1.3) \quad \|u\|_{2,1} \leq C(\log \text{rk}(u))^{1/2} \|u\|_2.$$

It is well known that the unit ball of $S_{2,1}(H)$ is equivalent (up to absolute constants) to the closed convex hull of the set formed by all operators of the form $u = k^{-1/2}P$ where P is a (orthogonal) projection of rank k . In particular, for any u we have (for some constant $C > 0$)

$$C^{-1} \|u\|_2 \leq \|u\|_{2,1}.$$

Therefore, if Z is a trilinear form on $S_{2,1}(H)$, we have (for some constant C)

$$(1.4) \quad \sup\{|Z(r, s, t)| \mid \|r\|_{2,1} \leq 1, \|s\|_{2,1} \leq 1, \|t\|_{2,1} \leq 1\} \leq C \sup_{k,l,m} (klm)^{-1/2} \sup |Z(R, S, T)|$$

where the second supremum runs over all integers k, l, m and the third one over all projections R, S, T of rank respectively k, l, m . Let us denote by $\mathcal{P}(k)$ the set of projections of rank k . By (1.3) letting $d = \dim(H)$, this implies (for some constant C)

$$(1.5) \quad \sup_{\|x\|_2 \leq 1, \|y\|_2 \leq 1, \|z\|_2 \leq 1} |Z(x, y, z)| \leq C(\log d)^{3/2} \sup_{k,l,m} (klm)^{-1/2} \sup_{(R,S,T) \in \mathcal{P}(k) \times \mathcal{P}(l) \times \mathcal{P}(m)} |Z(R, S, T)|.$$

Proof of Theorem. We identify $\ell_2^{N^2}$ with the Hilbert-Schmidt class $S_2^{N^2}$. Then, viewing R as an operator (or matrix) acting on ℓ_2^N , we denote by $\|R\|_2$ and $\|R\|_\infty$ respectively its Hilbert-Schmidt norm and operator norm. Let

$$Z(R, S, T) = \sum g_{i,k,l} g'_{i',k',l'} R_{ii'} S_{kk'} T_{ll'}$$

The norm of \mathcal{T} is the supremum of $Z(R, S, T)$ over R, S, T in the unit ball of $\ell_2^{N^2}$.

Note that $Z(R, S, T) = \langle (R \otimes S \otimes T)g, g' \rangle$ where g, g' are independent canonical random vectors on $\ell_2^{N^3}$, and $u = R \otimes S \otimes T$ is an operator on $\ell_2^{N^3} = \ell_2^N \otimes \ell_2^N \otimes \ell_2^N$.

Fix integers r, s, t and $\delta > 0$. Let $\mathcal{P}(r, s, t) = \mathcal{P}(r) \otimes \mathcal{P}(s) \otimes \mathcal{P}(t)$. Let $\mathcal{P}_\delta(r)$ be a δ -net in $\mathcal{P}(r)$ with respect to the norm in S_2 . It is easy to check that we can find such a net with at most $\exp\{c(\delta)rN\}$ elements, so we may assume that $|\mathcal{P}_\delta(r)| \leq \exp\{c(\delta)rN\}$. Let

$$\mathcal{P}_\delta(r, s, t) = \mathcal{P}_\delta(r) \otimes \mathcal{P}_\delta(s) \otimes \mathcal{P}_\delta(t).$$

Note that $\mathcal{P}_\delta(r, s, t)$ is a 3δ -net in $\mathcal{P}(r, s, t)$ and $|\mathcal{P}_\delta(r, s, t)| \leq \exp\{3c(\delta)(r + s + t)N\}$.

Let

$$\|Z\| = \sup_{\|x\|_2 \leq 1, \|y\|_2 \leq 1, \|z\|_2 \leq 1} |Z(x, y, z)|,$$

and

$$\|Z\|_\bullet = \sup_{\|x\|_{2,1} \leq 1, \|y\|_{2,1} \leq 1, \|z\|_{2,1} \leq 1} |Z(x, y, z)|.$$

Recall that by (1.5)

$$\|Z\|_\bullet \leq C(\log N)^{3/2} \|Z\|.$$

Claim: We claim that for some constant C_δ we have

$$\left\| \sup_{(R,S,T) \in \mathcal{P}_\delta(r,s,t)} (rst)^{-1/2} |Z(R, S, T)| \right\|_N \leq C_\delta N.$$

We will use a bound of Latała (actually easy to prove in the bilinear case) that says that for some absolute constant c we have for all $p \geq 1$

$$\|Z(R, S, T)\|_p \leq c(p^{1/2} \|R\|_2 \|S\|_2 \|T\|_2 + p \|R\|_\infty \|S\|_\infty \|T\|_\infty).$$

Let us record here for further reference the obvious inequality

$$(1.6) \quad \left\| \sup_{i \leq M} |Z(R_i, S_i, T_i)| \right\|_q \leq M^{1/q} \sup_{i \leq M} \|Z(R_i, S_i, T_i)\|_q.$$

When we take the sup over a family R_i, S_i, T_i indexed by $i = 1, \dots, M$ in the unit ball of S_2^N and such that for all i we have $\|R_i\|_\infty \leq r^{-1/2}$, $\|S_i\|_\infty \leq s^{-1/2}$, $\|T_i\|_\infty \leq t^{-1/2}$, this gives us

$$\left\| \sup_{i \leq M} |Z(R_i, S_i, T_i)| \right\|_p \leq cM^{1/p} (p^{1/2} + p(rst)^{-1/2}).$$

Choosing $p = \log M$ and $p \geq q$ we find a fortiori

$$(1.7) \quad \left\| \sup_{i \leq M} |Z(R_i, S_i, T_i)| \right\|_q \leq c'((\log M)^{1/2} + (\log M)(rst)^{-1/2}).$$

To prove the claim we may reduce to triples (r, s, t) such that $t = \max(r, s, t)$. Indeed exchanging the roles of (r, s, t) , we treat similarly the cases $r = \max(r, s, t)$ and $s = \max(r, s, t)$ and the desired result follows with a tripled constant C_δ .

We will treat separately the sets

$$A = \{(r, s, t) \mid rs > t, t = \max\{r, s\}\} \text{ and } B = \{(r, s, t) \mid rs \leq t, t = \max\{r, s\}\}.$$

Note that both sets have at most N^3 elements.

- Fix $(r, s, t) \in A$. By (1.7) with $q = N$, we have

$$\left\| \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} (rst)^{-1/2} |Z(R, S, T)| \right\|_N \leq c'((\log |\mathcal{P}_\delta(r, s, t)|)^{1/2} + (\log |\mathcal{P}_\delta(r, s, t)|))(rst)^{-1/2},$$

and hence

$$(1.8) \quad \leq c'(3c(\delta)(r + s + t)N)^{1/2} + c'(3c(\delta)(r + s + t)N)(rst)^{-1/2}.$$

Now on the one hand $(r + s + t)N^{1/2} \leq 3^{1/2}N$ and on the other hand, since we assume $t = \max\{r, s\}$ and $rs > t$, we have $(r + s + t)N(rst)^{-1/2} \leq 3tN(rst)^{-1/2} \leq 3N$. So assuming $(r, s, t) \in A$ we obtain

$$\left\| \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} (rst)^{-1/2} |Z(R, S, T)| \right\|_N \leq 3c'c(\delta)(3^{1/2} + 3)N.$$

- Now assume $(r, s, t) \in B$ and in particular $rs \leq t$. By Cauchy-Schwarz if R, S, T are projections we have $|Z(R, S, T)| = \langle g, (R \otimes S \otimes T)g' \rangle \leq \langle g, (R \otimes S \otimes T)g \rangle^{1/2} \langle g', (R \otimes S \otimes T)g' \rangle^{1/2}$. We may write a fortiori

$$\|Z(R, S, T)\|_p \leq \|\langle g, (R \otimes S \otimes T)g \rangle\|_p.$$

Therefore by (1.2) and since $T \leq I$ (and hence $R \otimes S \otimes T \leq R \otimes S \otimes I$) we have

$$\|Z(R, S, T)\|_p \leq 2\|Z(R, S, I)\|_p + rsN.$$

Similarly we find

$$(1.9) \quad \left\| \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} |Z(R, S, T)| \right\|_p \leq 2 \left\| \sup_{(R, S, I) \in \mathcal{P}_\delta(r, s, N)} |Z(R, S, I)| \right\|_p + rsN,$$

and hence by (1.7) again (we argue as for (1.8) above with $q = N$, but note however that in the present case $T = I$ is fixed so the supremum over $(R, S, I) \in \mathcal{P}_\delta(r, s, N)$ runs over at most $\exp\{3c(\delta)(r + s)N\}$ elements and we may use $p = 3c(\delta)(r + s)N$) we find

$$\left\| \sup_{(R, S, I) \in \mathcal{P}_\delta(r, s, N)} (rst)^{-1/2} |Z(R, S, I)| \right\|_N \leq c'(3c(\delta)(r + s)N)^{1/2} (N/t)^{1/2} + c'(3c(\delta)(r + s)N)(rst)^{-1/2}.$$

But now, since we assume $rs \leq t$, $r + s \leq 2rs \leq 2t$ so that $(r + s)(rst)^{-1/2} \leq 2(rs/t)^{1/2} \leq 2$, and hence we find $\left\| \sup_{(R, S, I) \in \mathcal{P}_\delta(r, s, N)} (rst)^{-1/2} |Z(R, S, I)| \right\|_N \leq c_3(\delta)N$. Substituting this in (1.9) yields

$$\left\| \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} (rst)^{-1/2} |Z(R, S, T)| \right\|_N \leq 2c_3(\delta)N + (rs/t)^{1/2}N \leq (2c_3(\delta) + 1)N.$$

This completes the proof of the above claim.

Using the claim, we conclude the proof as follows: To pass from $\mathcal{P}_\delta(r, s, t)$ to $\mathcal{P}(r, s, t)$ we first note that if (say) P, P' are both projections of rank r , by (1.3) $\|P - P'\|_2 \leq \delta$ implies (for some constant c_1) that $r^{-1/2}\|P - P'\|_{2,1} \leq c_1\delta$. Thus, using this for r, s, t successively, we find

$$\sup_{(R, S, T) \in \mathcal{P}(r, s, t)} (rst)^{-1/2} |Z(R, S, T)| \leq \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} (rst)^{-1/2} |Z(R, S, T)| + 3c_1\delta\|Z\|_\bullet.$$

This implies

$$\left\| \sup_{(R,S,T) \in \mathcal{P}(r,s,t)} (rst)^{-1/2} |Z(R,S,T)| \right\|_N \leq \left\| \sup_{(R,S,T) \in \mathcal{P}_\delta(r,s,t)} (rst)^{-1/2} |Z(R,S,T)| \right\|_N + 3c_1 \delta \left\| \|Z\|_\bullet \right\|_N.$$

Using (1.6) to estimate the sup over the N^3 intergers r, s, t we find (recalling (1.4))

$$\left\| \|Z\|_\bullet \right\|_N \leq CN^{3/N} \sup_{r,s,t} \left(\left\| \sup_{(R,S,T) \in \mathcal{P}_\delta(r,s,t)} (rst)^{-1/2} |Z(R,S,T)| \right\|_N + 3c_1 \delta \left\| \|Z\|_\bullet \right\|_N \right),$$

and hence by the claim

$$\leq 8CC_\delta N + 24C_1 \delta \left\| \|Z\|_\bullet \right\|_N$$

from which follows that

$$\left\| \|Z\|_\bullet \right\|_N \leq (1 - 24c_1 C \delta)^{-1} 8CC_\delta N.$$

Observe that $\mathbb{E}\|Z\| \leq (\log N)^{3/2} \mathbb{E}\|Z\|_\bullet \leq (\log N)^{3/2} \left\| \|Z\|_\bullet \right\|_N$. Thus, if δ is small enough, chosen so that say $24c_1 C \delta = 1/2$, we finally obtain a fortiori

$$\mathbb{E}\|Z\| \leq 16CC_\delta (\log N)^{3/2} N.$$

Actually since we obtain the same bound for $(\mathbb{E}\|Z\|^N)^{1/N}$ we also obtain for suitable positive constants c_2, c_3 that

$$\mathbb{P}\{\|Z\| > c_3 N (\log N)^{3/2}\} \leq \exp -c_2 N.$$

□

The Theorem has the following application improving a result in Junge etal [4]:

Theorem 1.5. *Consider the following two norms for an element $t = \sum_{ijk} t_{ijk} e_i \otimes e_j \otimes e_k$ in the triple tensor product $\ell_1^n \otimes \ell_1^n \otimes \ell_1^n$:*

$$(1.10) \quad \|t\|_{\min} = \sup\left\{ \left\| \sum_{ijk} t_{ijk} u_i \otimes v_j \otimes w_k \right\|_{B(H \otimes_2 H \otimes_2 H)} \right\}$$

where the sup runs over all possible Hilbert spaces H and all possible unitary operators u_i, v_j, w_k acting on H , and also:

$$(1.11) \quad \|t\|_{\vee} = \sup\left\{ \left| \sum_{ijk} t_{ijk} x_i y_j z_k \right| \right\}$$

where the sup runs over all unimodular scalars x_i, y_j, z_k or equivalently the sup is as before but restricted to $\dim(H) = 1$. Let

$$(1.12) \quad C_3(n) = \sup\{\|t\|_{\min} \mid \|t\|_{\vee} \leq 1\}.$$

Then we have for some constant $C' > 0$ (independent of n)

$$C_3(n) \geq C' n^{1/4} (\log n)^{-3/2}.$$

Remark 1.6. It is well known that the supremum in (1.10) is unchanged if we restrict the supremum to finite dimensional spaces H . Moreover, we have also

$$\|t\|_{\min} = \sup\left\{ \left\| \sum_{ijk} t_{ijk} u_i v_j w_k \right\|_{M_N} \right\},$$

where the supremum runs over all N and all $N \times N$ -unitary matrices u_i, v_j, w_k such that $u_i v_j = v_j u_i$, $u_i w_k = w_k u_i$ and $w_k v_j = v_j w_k$ for all i, j, k .

Note that (1.11) corresponds again to restricting this sup to $N = 1$.

Proof of Theorem 1.5. Let $n = N^2$. We again identify $\ell_2^{N^2}$ with the space of $N \times N$ matrices equipped with the HS norm. Let $\{u_j \mid j \leq N^2\}$ be an orthogonal basis in $\ell_2^{N^2}$ consisting of unitaries (this is called an EPR basis in quantum information). Note that $\|u_j\|_2 = \sqrt{N}$ for all j . Then $u_i \otimes u_j \otimes u_k$ ($i, j, k \leq N^2$) forms an orthogonal basis in $\ell_2^{N^2} \otimes \ell_2^{N^2} \otimes \ell_2^{N^2}$.

Consider now $T \in \ell_2^{N^2} \otimes \ell_2^{N^2} \otimes \ell_2^{N^2}$ and let

$$T = \sum \hat{T}_{ijk} u_i \otimes u_j \otimes u_k$$

be its development on that orthogonal basis, so that $\hat{T}_{ijk} = N^{-3} \langle T, u_i \otimes u_j \otimes u_k \rangle$. Consider now $t = \sum_{ijk} \hat{T}_{ijk} e_i \otimes e_j \otimes e_k$. Then for any unimodular scalars x_i, y_j, z_k we have

$$\sum \hat{T}_{ijk} x_i \otimes y_j \otimes z_k = \langle T, X \otimes Y \otimes Z \rangle$$

with $X = \sum x_i u_i$, $Y = \sum y_j u_j$, $Z = \sum z_k u_k$ and hence

$$\|t\|_v \leq N^{3/2} \sup\{|\langle T, X \otimes Y \otimes Z \rangle| \mid X, Y, Z \in B_{\ell_2^{N^2}}\}.$$

But since we also have $T = \sum \hat{T}_{ijk} u_i \otimes u_j \otimes u_k$, we have

$$\|t\|_{\min} \geq \|T\|_{B(\ell_2^N \otimes \ell_2^N \otimes \ell_2^N)}.$$

Now consider as before

$$T = \sum g_{i,k,l} g'_{i',k',l'} e_{ii'} \otimes e_{kk'} \otimes e_{ll'}.$$

With this choice of T by the preceding Theorem we find with large probability $\sup\{|\langle T, X \otimes Y \otimes Z \rangle| \mid X, Y, Z \in B_{\ell_2^{N^2}}\} \leq CN(\log N)^{3/2}$, and hence

$$\|t\|_v \leq CN^{5/2}(\log N)^{3/2},$$

but also (since T is a rank one operator) $\|T\|_{B(\ell_2^N \otimes \ell_2^N \otimes \ell_2^N)} = (\sum |g_{i,k,l}|^2)^{1/2} (\sum |g'_{i',k',l'}|^2)^{1/2}$ and the latter is concentrated around its mean and hence with large probability $\geq N^3/2$. Thus we conclude that

$$C(N^2) \geq (2C)^{-1} N^{1/2} (\log N)^{-3/2}.$$

□

Remark 1.7. The same method works in the d -linear case. Consider

$$T = \sum g_{i(1), \dots, i(d)} g'_{i'(1), \dots, i'(d)} e_{i(1)i'(1)} \otimes \dots \otimes e_{i(d)i'(d)} \in \ell_1^{N(1)^2} \otimes \dots \otimes \ell_1^{N(d)^2}.$$

Let $\{u_i^{(j)} \mid 1 \leq i \leq N(j)^2\}$ be an orthogonal basis in $S_2^{N(j)}$ formed of unitary matrices. We will denote by \underline{i} the elements of the set $\underline{I} = [N(1)^2 \times \dots \times N(d)^2]$. Let

$$T = \sum \hat{T}(\underline{i}) u_{\underline{i}}$$

be its orthogonal development according to the basis formed by

$$\forall \underline{i} = (i(1), \dots, i(d)) \in \underline{I} \quad u_{\underline{i}} = u_{i(1)}^{(1)} \otimes \dots \otimes u_{i(d)}^{(d)},$$

so that

$$\hat{T}(\underline{i}) = (N(1) \dots N(d))^{-1} \langle T, u_{\underline{i}} \rangle.$$

Let us denote also by $e_{\underline{i}} = e_{i(1)} \otimes \cdots \otimes e_{i(d)}$ the canonical basis in $\ell_1^{N(1)^2} \otimes \cdots \otimes \ell_1^{N(d)^2}$. Let then

$$t = \sum \hat{T}(\underline{i}) e_{\underline{i}}.$$

As above, on one hand since T appears as an operator of rank one, we have

$$\|t\|_{\min} \geq \|T\|_{B(\ell_2^{N(1)} \otimes \cdots \otimes \ell_2^{N(d)})} \geq \left(\sum |g_{i(1), \dots, i(d)}|^2 \right)^{1/2} \left(\sum |g'_{i(1), \dots, i(d)}|^2 \right)^{1/2}.$$

On the other hand using the orthogonality of the $u_{\underline{i}}$'s we have

$$\|t\|_{\vee} \leq (N(1) \cdots N(d))^{1/2} \|T\|_{\ell_2^{N(1)^2} \otimes \cdots \otimes \ell_2^{N(d)^2}}.$$

Thus we find

$$C_d(N(1)^2, \dots, N(d)^2) \geq (N(1) \cdots N(d))^{-1/2} \left(\sup \|T\|_{\ell_2^{N(1)^2} \otimes \cdots \otimes \ell_2^{N(d)^2}} \right)^{-1},$$

where the sup runs over all T of the above form (i.e. of rank one in a suitable sense) such that $(\sum |g_{i(1), \dots, i(d)}|^2)^{1/2} (\sum |g'_{i(1), \dots, i(d)}|^2)^{1/2} \leq 1$ (i.e. of norm one in a suitable sense). Then using Gaussian variables as above, we obtain $\|t\|_{\min} \geq cN^d$ and $\|t\|_{\vee} \leq cN^{d/2} (N(\log N)^{d/2})$

$$C_d(N^2, \dots, N^2) \geq c_d N^{d/2-1} (\log N)^{-d/2}.$$

2. An almost sharp inequality

Let (H_j) and (K_j) ($1 \leq j \leq d$) be d -tuples of finite dimensional Hilbert spaces. Let

$$J : (H_1 \otimes_2 K_1) \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} (H_d \otimes_2 K_d) \rightarrow (H_1 \otimes_2 \cdots \otimes_2 H_d) \overset{\vee}{\otimes} (K_1 \otimes_2 \cdots \otimes_2 K_d),$$

be the natural identification map. After reordering, we may as well assume that the sequence $\{\dim(H_j) \dim(K_j) \mid 1 \leq j \leq d\}$ is non-decreasing.

We have

$$(2.1) \quad \|J\| \leq \prod_{j=1}^{d-1} (\dim(H_j) \dim(K_j))^{1/2}.$$

Indeed, it is easy to check that, for any normed space E , the identity map $\ell_2^n \overset{\vee}{\otimes} E \rightarrow \ell_2^n(E)$ has norm $\leq \sqrt{n}$. This gives

$$\|(H_1 \otimes_2 K_1) \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} (H_d \otimes_2 K_d) \rightarrow (H_1 \otimes_2 K_1) \otimes_2 \cdots \otimes_2 (H_d \otimes_2 K_d)\| \leq \prod_{j=1}^{d-1} (\dim(H_j) \dim(K_j))^{1/2}.$$

A fortiori we obtain the above bound (2.1) for J .

Consider now the case when $\dim(H_j) = \dim(K_j) = N$ for all j . In that case the preceding bound becomes

$$\|J\| \leq N^{d-1}.$$

Consider again

$$T = \sum g_{i(1), \dots, i(d)} g'_{i'(1), \dots, i'(d)} e_{i(1)i'(1)} \otimes \cdots \otimes e_{i(d)i'(d)} \in \ell_2^{N^2} \otimes \cdots \otimes \ell_2^{N^2},$$

where we identify $H_j \otimes_2 K_j$ with $\ell_2^{N^2}$. The preceding proof yields with large probability

$$\|T\|_{\ell_2^{N^2} \otimes \dots \otimes \ell_2^{N^2}} \leq cN(\log N)^{d/2}$$

and also

$$\|T\|_{(H_1 \otimes_2 \dots \otimes_2 H_d) \otimes (K_1 \otimes_2 \dots \otimes_2 K_d)} \geq c'N^d.$$

Thus we obtain the following almost sharp (i.e. sharp up to the log factor)

$$\|J\| \geq c''N^{d-1}(\log N)^{-d/2}.$$

The argument described in the preceding Remark 1.7 boils down to the estimate

$$C_d(N^2, \dots, N^2) \geq N^{-d/2}\|J\|.$$

Remark 2.1. Note that the preceding proof actually yields (assuming $\dim(H_j) = \dim(K_j) = N$ for all j)

$$c''N^{d-1}(\log N)^{-d/2} \leq \|(H_1 \otimes_2 K_1) \otimes \dots \otimes (H_d \otimes_2 K_d) \rightarrow (H_1 \otimes_2 K_1) \otimes_2 \dots \otimes_2 (H_d \otimes_2 K_d)\| \leq N^{d-1}.$$

However, this norm is much easier to estimate and, actually, we claim it is $\geq c_d N^{d-1}$.

Indeed, returning to H_j, K_j of arbitrary finite dimension, let $n_j = \dim(H_j) \dim(K_j)$.

Assume $n_1 \leq n_2 \leq \dots \leq n_d$. Consider then the inclusion

$$\Phi : \ell_2^{n_1} \otimes \dots \otimes \ell_2^{n_d} \rightarrow \ell_2^{n_1 \dots n_d}.$$

The above easy argument shows that $\|\Phi\| \leq (n_1 \dots n_{d-1})^{1/2}$. Let now G be a random vector with values in $\ell_2^{n_1 \dots n_d}$ distributed according to the canonical Gaussian measure. We will identify G with $\Phi^{-1}(G)$. Then, by Simone Chevet's well known inequality we have

$$\mathbb{E}\|G\|_{\ell_2^{n_1} \otimes \dots \otimes \ell_2^{n_d}} \leq \sqrt{d} \sum_j \sqrt{n_j} \leq d^{3/2} n_d,$$

while it is clear that $\mathbb{E}\|G\|_{\ell_2^{n_1 \dots n_d}}^2 = n_1 \dots n_d$. From this follows that $\|\Phi\| \geq d^{-3/2}(n_1 \dots n_{d-1})^{1/2}$. In particular the above claim is established.

3. A different method

It is known (due to Geman) that

$$(3.1) \quad \lim_{N \rightarrow \infty} \|Y^{(N)}\|_{M_N} = 2 \quad \text{a.s.}$$

Let $(Y_1^{(N)}, Y_2^{(N)}, \dots)$ be a sequence of independent copies of $Y^{(N)}$, so that the family $\{Y_k^N(i, j) \mid k \geq 1, 1 \leq i, j \leq N\}$ is an independent family of $N(0, N^{-1})$ complex Gaussian.

The next two statements follow from results known to Steen Thorbjørnsen since at least 1999 (private communication). See [2] for closely related results. We present a trick that yields a self-contained derivation of this.

Theorem 3.1. Consider independent copies $Y'_i = Y_i^{(N)}(\omega')$ and $Y''_j = Y_j^{(N)}(\omega'')$ for $(\omega', \omega'') \in \Omega \times \Omega$. Then, for any n^2 -tuple of scalars (α_{ij}) , we have

$$(3.2) \quad \overline{\lim}_{N \rightarrow \infty} \left\| \sum \alpha_{ij} Y_i^{(N)}(\omega') \otimes Y_j^{(N)}(\omega'') \right\|_{M_{N^2}} \leq 4 \left(\sum |\alpha_{ij}|^2 \right)^{1/2}$$

for a.e. (ω', ω'') in $\Omega \times \Omega$.

Proof. By (well known) concentration of measure arguments, it is known that (3.1) is essentially the same as the assertion that $\lim_{N \rightarrow \infty} \mathbb{E} \|Y^{(N)}\|_{M_N} = 2$. Let $\varepsilon(N)$ be defined by

$$\mathbb{E} \|Y^{(N)}\|_{M_N} = 2 + \varepsilon(N)$$

so that we know $\varepsilon(N) \rightarrow 0$. Again by concentration of measure arguments (see e.g. [3, p. 41] or [5, (1.4) or chapter 2]) there is a constant β such that for any $N \geq 1$ and $p \geq 2$ we have

$$(3.3) \quad (\mathbb{E} \|Y^{(N)}\|_{M_N}^p)^{1/p} \leq \mathbb{E} \|Y^{(N)}\|_{M_N} + \beta(p/N)^{1/2} \leq 2 + \varepsilon(N) + \beta(p/N)^{1/2}.$$

For any $\alpha \in M_n$, we denote

$$Z^{(N)}(\alpha)(\omega', \omega'') = \sum_{i,j=1}^n \alpha_{ij} Y_i^{(N)}(\omega') \otimes Y_j^{(N)}(\omega'').$$

Assume $\sum_{ij} |\alpha_{ij}|^2 = 1$. We will show that almost surely

$$\lim_{N \rightarrow \infty} \|Z^{(N)}(\alpha)\| \leq 4.$$

Note that by the invariance of (complex) canonical Gaussian measures under unitary transformations, $Z^{(N)}(\alpha)$ has the same distribution as $Z^{(N)}(u\alpha v)$ for any pair u, v of $n \times n$ unitary matrices. Therefore, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $|\alpha| = (\alpha^* \alpha)^{1/2}$, we have

$$Z^{(N)}(\alpha)(\omega', \omega'') \stackrel{\text{dist}}{=} \sum_{j=1}^n \lambda_j Y_j^{(N)}(\omega') \otimes Y_j^{(N)}(\omega'').$$

We claim that by a rather simple calculation of moments, one can show that for any even integer $p \geq 2$ we have

$$(3.4) \quad \mathbb{E} \operatorname{tr} |Z^{(N)}(\alpha)|^p \leq (\mathbb{E} \operatorname{tr} |Y^{(N)}|^p)^2.$$

Accepting this claim for the moment, we find, a fortiori, using (3.3):

$$\mathbb{E} \|Z^{(N)}(\alpha)\|_{M_N}^p \leq N^2 (\mathbb{E} \|Y^{(N)}\|_{M_N}^p)^2 \leq N^2 (2 + \varepsilon(N) + \beta(p/N)^{1/2})^{2p}.$$

Therefore for any $\delta > 0$

$$\mathbb{P}\{\|Z^{(N)}(\alpha)\|_{M_N} > (1 + \delta)4\} \leq (1 + \delta)^{-p} N^2 (1 + \varepsilon(N)/2 + (\beta/2)(p/N)^{1/2})^{2p}.$$

Then choosing (say) $p = 5(1/\delta) \log(N)$ we find

$$\mathbb{P}\{\|Z^{(N)}(\alpha)\|_{M_N} > (1 + \delta)4\} \in O(N^{-2})$$

and hence (Borel–Cantelli) $\overline{\lim}_{N \rightarrow \infty} \|Z^{(N)}(\alpha)\|_{M_N} \leq 4$ a.s..

It remains to verify the claim. Let $Z = Z^{(N)}(\alpha)$, $Y = Y^{(N)}$ and $p = 2m$. We have

$$\mathbb{E} \operatorname{tr} |Z|^p = \mathbb{E} \operatorname{tr} (Z^* Z)^m = \sum \bar{\lambda}_{i_1} \lambda_{j_1} \dots \bar{\lambda}_{i_m} \lambda_{j_m} (\mathbb{E} \operatorname{tr} (Y_{i_1}^* Y_{j_1} \dots Y_{i_m}^* Y_{j_m}))^2.$$

Note that the only nonvanishing terms in this sum correspond to certain pairings that guarantee that both $\bar{\lambda}_{i_1} \lambda_{j_1} \dots \bar{\lambda}_{i_m} \lambda_{j_m} \geq 0$ and $\mathbb{E} \operatorname{tr}(Y_{i_1}^* Y_{j_1} \dots Y_{i_m}^* Y_{j_m}) \geq 0$. Moreover, by Hölder's inequality for the trace we have

$$|\mathbb{E} \operatorname{tr}(Y_{i_1}^* Y_{j_1} \dots Y_{i_m}^* Y_{j_m})| \leq \Pi(\mathbb{E} \operatorname{tr}|Y_{i_k}|^p)^{1/p} \Pi(\mathbb{E} \operatorname{tr}|Y_{j_k}|^p)^{1/p} = \mathbb{E} \operatorname{tr}(|Y|^p).$$

From these observations, we find

$$(3.5) \quad \mathbb{E} \operatorname{tr}|Z|^p \leq \mathbb{E} \operatorname{tr}(|Y|^p) \sum \bar{\lambda}_{i_1} \lambda_{j_1} \dots \bar{\lambda}_{i_m} \lambda_{j_m} \mathbb{E} \operatorname{tr}(Y_{i_1}^* Y_{j_1} \dots Y_{i_m}^* Y_{j_m})$$

but the last sum is equal to $\mathbb{E} \operatorname{tr}(|\sum \lambda_j Y_j|^p)$ and since $\sum \lambda_j Y_j \stackrel{\text{dist}}{=} Y$ (recall $\sum |\lambda_j|^2 = \sum |\alpha_{ij}|^2 = 1$) we have

$$\mathbb{E} \operatorname{tr}\left(\left|\sum \alpha_j Y_j\right|^p\right) = \mathbb{E} \operatorname{tr}(|Y|^p),$$

and hence (3.5) implies (3.4). \square

Corollary 3.2. *For any integer n and $\varepsilon > 0$, there are N and n -tuples of $N \times N$ matrices $\{Y'_i \mid 1 \leq i \leq n\}$ and $\{Y''_j \mid 1 \leq j \leq n\}$ in M_N such that*

$$(3.6) \quad \sup \left\{ \left\| \sum_{i,j=1}^n \alpha_{ij} Y'_i \otimes Y''_j \right\|_{M_{N^2}} \mid \alpha_{ij} \in \mathbb{C}, \sum_{ij} |\alpha_{ij}|^2 \leq 1 \right\} \leq (4 + \varepsilon)$$

$$(3.7) \quad \min \left\{ \frac{1}{nN} \sum_1^n \operatorname{tr}|Y'_i|^2, \frac{1}{nN} \sum_1^n \operatorname{tr}|Y''_j|^2 \right\} \geq 1 - \varepsilon.$$

Proof. Fix $\varepsilon > 0$. Let \mathcal{N}_ε be a finite ε -net in the unit ball of $\ell_2^{n^2}$. By Theorem 3.1 we have for almost all (ω', ω'')

$$(3.8) \quad \overline{\lim}_{N \rightarrow \infty} \sup_{\alpha \in \mathcal{N}_\varepsilon} \left\| \sum_{i,j=1}^n \alpha_{ij} Y'_i \otimes Y''_j \right\|_{M_{N^2}} \leq 4,$$

We may pass from an ε -net to the whole unit ball in (3.8) at the cost of an extra factor $(1 + \varepsilon)$ and we obtain (3.6). As for (3.7), the strong law of large numbers shows that the left side of (3.7) tends a.s. to 1. Therefore, we may clearly find (ω', ω'') satisfying both (3.6) and (3.7). \square

Remark 3.3. A close examination of the proof and concentration of measure arguments show that the preceding corollary holds with N of the order of $c(\varepsilon)n^2$. Indeed, we find a constant C such that for any $\alpha = (\alpha_{ij})$ in the unit ball of $\ell_2^{n^2}$ we have (we take $p = N$)

$$\|Z^{(N)}(\alpha)\|_{L_N(M_N)} \leq C$$

from which follows if A is a finite subset of the unit ball of $\ell_2^{n^2}$ that

$$\left\| \sup_{\alpha \in A} \|Z^{(N)}(\alpha)\|_{M_N} \right\|_N \leq C|A|^{1/N}.$$

So if we choose for A an ε -net in the unit ball of $\ell_2^{n^2}$ with $|A| \approx 2^{cn^2}$, and if $N = n^2$ we still obtain a fortiori

$$\left\| \sup_{\alpha \in A} \|Z^{(N)}(\alpha)\|_{M_N} \right\|_1 \leq C'.$$

Remark 3.4. Using the well known “contraction principle” that says that the variables (ε_j) are dominated by either $(g_j^{\mathbb{R}})$ or $(g_j^{\mathbb{C}})$, it is easy to deduce that Corollary 3.2 is valid for matrices Y'_i, Y''_j with entries all equal to $\pm N^{-1/2}$, with possibly a different numerical constant in place of 4. Analogously, using the polar factorizations $Y'_i = U'_i |Y'_i|$, $Y''_j = U''_j |Y''_j|$ and noting that all the factors $U'_i, |Y'_i|, U''_j, |Y''_j|$ are independent, we can also (roughly by integrating over the moduli $|Y'_i|, |Y''_j|$) obtain Corollary 3.2 with unitary matrices Y'_i, Y''_j , with a different numerical constant in place of 4.

Let $C_3(n_1, n_2, n_3)$ the supremum appearing in (1.12) when t runs over all tensors in $\ell_1^{n_1} \otimes \ell_1^{n_2} \otimes \ell_1^{n_3}$. Note that $C_3(n) = C_3(n, n, n)$. Then the proof of Junge et al as presented in [7] (and incorporating the results of [6]) yields

$$C_3(n^4, n^8, n^8) \geq cn^{1/2}.$$

Indeed, the latter proof requires an embedding of $\ell_2^{n^2}$ into ℓ_1^m and Junge et al use the Rademacher embedding, and hence $m = 2^{n^2}$, but [6] allows us to use $m = n^4$.

However, if we use the method of Theorem 1.1 together with Corollary 3.2 and the estimate in Remark 3.3, then we find

$$C_3(n^2, n^4, n^4) \geq cn^{1/2}.$$

The open problems that remain are:

- get rid of the log factor in Theorem 1.1.
- improve the lower bound of $C_3(n)$ in Theorem 1.5 to something sharp, possibly $cn^{1/2}$, and similar questions for $C_3(n_1, n_2, n_3)$.
- find explicit non random examples responsible for large values of $C_3(n)$.

4. Upper bounds

As far as I know the upperbounds for $C_3(n)$ or $C_d(n)$ are as follows.

First we have $C_2(n) \leq K_G$ (here K_G is the Grothendieck constant).

If E is any operator space, let $\min(E)$ be the same Banach space but viewed as embedded in a commutative C^* -algebra (i.e. the continuous functions on the dual unit ball).

Let F be an arbitrary operator space, it is easy to show that we have isometric identities

$$F \otimes_{\min} \min(E) = F \overset{\vee}{\otimes} E = \min(F) \otimes_{\min} \min(E).$$

Moreover, it is known (and easy to check) that if $E = \ell_1^n$, the identity map $\min(E) \rightarrow E$ has cb norm at most \sqrt{n} . A fortiori, we have $\|F \otimes_{\min} \min(E) \rightarrow F \otimes_{\min} E\|_{cb} \leq \sqrt{n}$ and hence

$$\|F \overset{\vee}{\otimes} E \rightarrow F \otimes_{\min} E\| \leq \sqrt{n}.$$

This implies that if $E_d = \ell_1^n \otimes_{\min} \cdots \otimes_{\min} \ell_1^n$ (d times), then

$$\|E_{d-1} \overset{\vee}{\otimes} E \rightarrow E_d\| \leq \sqrt{n}.$$

Iterating we find

$$\|E \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} E \rightarrow E_d\| \leq C_{d-1}(n) \sqrt{n}.$$

Thus we obtain

$$C_d(n) \leq K_G n^{(d-2)/2}.$$

A similar argument yields (note that we can use invariance under permutation to reduce to the case when $n_1 \leq n_2 \leq \cdots$) whenever $d \geq 3$:

$$C_d(n_1, \dots, n_d) \leq K_G (n_1 \cdots n_{d-2})^{1/2}.$$

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